7.2 Laplace Transform of Initial-Value Problems

solve something like $\underbrace{y''+4y=8}$ $y(0)=0$, $y'(0)=6$ ← initial velocity

initial displacement

mass = 1

spring constant = 4

force of (upward) 8

basic idea: Laplace transform of both sides

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y'(t)\} = ?$

$\mathcal{L}\{y''(t)\} = ?$

$$\mathcal{L}\{y'\} = \int_0^\infty y' e^{-st}\,dt = \lim_{a\to\infty}\int_0^a y' e^{-st}\,dt$$

$u = e^{-st}$ $\quad dv = y'\,dt$

$du = -se^{-st}\,dt$ $\quad v = y$

$$= \lim_{a\to\infty} \left(y(t)e^{-st} \Big|_0^a + s\int_0^a y e^{-st}\,dt \right)$$

$$= \lim_{a\to\infty} \underbrace{y(a)e^{-sa}}_{\substack{\to 0 \\ s>0}} - y(0) + s \underbrace{\int_0^\infty y e^{-st}\,dt}_{\mathcal{L}\{y\} = Y}$$

$$\boxed{\mathcal{L}\{y'\} = sY - y(0)}$$

likewise, $\boxed{\mathcal{L}\{y''\} = s^2 Y - sy(0) - y'(0)}$

$$\mathcal{L}\{y^{(n)}\} = s^n Y - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0)$$

now back to $y'' + 4y = 8 \qquad y(0) = 0,\ y'(0) = 6$

Laplace transform: $\mathcal{L}\{y''\} + \mathcal{L}\{4y\} = \mathcal{L}\{8\}$

$$s^2 Y - s\underset{\uparrow \atop 0}{y(0)} - \underset{\uparrow \atop 6}{y'(0)} + 4Y = \frac{8}{s}$$

solve for Y : $(s^2+4)Y = 6 + \frac{8}{s}$

$$Y = \frac{6}{s^2+4} + \frac{8}{s(s^2+4)}$$

problem "solved"
but we would like to have
$y(t)$

inverse Laplace transform to find $y(t)$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\}$$

$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin(at)$ from table

$$\mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\} = 6\,\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

want a 2 here

$$= 3\,\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\}$$

$$= 3\sin(2t)$$

$\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\}$ is a bit more complicated

table entries are all $\frac{1}{s-a}$ or $\frac{1}{s^2 \pm a^2}$ no $\frac{1}{s(s^2 \pm a^2)}$

partial fraction expansion: $\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}$ ← linear (one degree lower than denom) ← quadratic

$$8 = A(s^2+4) + (Bs+C)s$$

$$0s^2 + 0s + 8 = (A+B)s^2 + Cs + 4A$$

$$A+B=0$$

$$C=0$$

$$4A = 8 \quad \text{so } A=2, \; B=-2$$

so, $\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s} - \frac{2s}{s^2+4}\right\}$

$$= 2 - 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} = 2 - 2\cos(2t)$$

so, $\boxed{y(t) = 3\sin(2t) + 2 - 2\cos(2t)}$

In some cases, there is an alternative to doing partial fraction expansion

we know $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

what about $\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = ?$

let $g(t) = \int_0^t f(\tau)\,d\tau$

then from calculus,

$$g'(t) = f(t)$$

$$\mathcal{L}\{g'(t)\} = sG - \underbrace{g(0)}_{0} = sG$$

~~so, $\mathcal{L}\{$~~

$$\mathcal{L}\{f(t)\} = sG = s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$$

or $\quad F = s\,\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\}$

or $\quad \dfrac{F}{s} = \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} \rightarrow \boxed{\mathcal{L}^{-1}\left\{\dfrac{F}{s}\right\} = \int_0^t f(\tau)\,d\tau}$

$$\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{\frac{8}{s^2+4}}{s}\right\}$$

find $\underbrace{\mathcal{L}^{-1}\left\{\frac{8}{s^2+4}\right\}}_{4\sin(2t)}$ then integrate

$$= \int_0^t 4\sin(2\tau)\,d\tau = -2\cos(2\tau)\Big|_0^t = -2\cos(2t)+2$$